

# Proper forcing and $L(\mathbb{R})$

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## Abstract

We present two ways in which the model  $L(\mathbb{R})$  is canonical assuming the existence of large cardinals. We show that the theory of this model, with *ordinal* parameters, cannot be changed by small forcing; we show further that a set of ordinals in  $V$  cannot be added to  $L(\mathbb{R})$  by small forcing. The large cardinal needed corresponds to the consistency strength of  $AD^{L(\mathbb{R})}$ ; roughly  $\omega$  Woodin cardinals.

## 1 Introduction

It is well known that under the existence of large cardinals the theory of  $L(\mathbb{R})$  —possibly with real parameters— is absolute, and in particular cannot be changed by small forcings. Things may be different if one considers the theory of  $L(\mathbb{R})$  with *ordinal* parameters. By results of Woodin and Shelah this theory can be changed by semiproper forcing, even granted large cardinals. In fact the truth value of the formula  $\phi(\tilde{\alpha}) = “\tilde{\alpha} \text{ is equal to } \omega_2”$  can easily be changed as follows: Start with  $V$  having, e.g., a supercompact. Then  $L(\mathbb{R}^V) \models \phi[\alpha]$  for some  $\alpha \leq \omega_2^V$ , simply because any cardinal of  $V$  must be a cardinal of  $L(\mathbb{R}^V)$ . Using the supercompact force to make the SemiProper Forcing Axiom hold in the generic extension. The supercompact of  $V$  becomes  $\omega_2$  of  $V[G]$ , and in  $V[G]$  every set still has a sharp. All this can be done with a semiproper forcing. From SPFA it follows in  $V[G]$  that the non-stationary ideal on  $\omega_1$  is saturated (see [Jec87]; this result is due to Shelah), and by results of Woodin this (with sharps) implies that  $\omega_2$  as computed in  $L(\mathbb{R}^{V[G]})$  is equal to  $\omega_2^{V[G]}$  (see [Woo99]). Now this is greater than  $\omega_2^V$ , and so certainly greater than  $\alpha$ . Thus  $L(\mathbb{R}^V) \models \phi[\alpha]$  while  $L(\mathbb{R}^{V[G]}) \not\models \phi[\alpha]$ .

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This example demonstrates that semiproper forcings *can* change the theory of  $L(\mathbb{R})$  with ordinal parameters greater than or equal to  $\omega_2$ .<sup>1</sup> Any attempt to prove the preservation of this theory must therefore be restricted to a class of forcing strictly smaller than semiproper.

**Theorem 1 (Embedding Theorem)** *(Under large cardinal assumption  $A_\kappa$ , see below.) Let  $P$  be a proper forcing notion of size  $\leq \kappa$ , and let  $G$  be  $P$ -generic/ $V$ . Then there exists an elementary embedding  $j: L(\mathbb{R}^V) \rightarrow L(\mathbb{R}^{V[G]})$  which is the identity on all ordinals.*

The large cardinal assumption,  $A_\kappa$ , of Theorem 1 is the following:

( $A_\kappa$ ) There exists a class inner model  $M$  and a countable ordinal  $\delta$  so that

- $M = L(V_\delta^M)$ ;
- $M \models$  “ $\delta$  is the supremum of  $\omega$  Woodin cardinals;” and
- $M$  is uniquely iterable for iteration trees of length  $\leq \kappa^+$ .

*Uniquely iterable* means basically that in the course of our proof we are free to create iteration trees on  $M$ , without having to worry about the existence of cofinal well-founded branches. More precisely  $M$  must be iterable, meaning that the good player must win the full iteration game of [MS94] on  $M$  of length  $\kappa^+ + 1$ . Furthermore the choice of cofinal branches must be unique, in the sense that for any iteration tree on  $M$  of size  $< \kappa^+$  there must be a *unique* cofinal branch  $b$  such that the direct limit model  $M_b$  is itself iterable. The technical assumption  $A_\kappa$  is weaker than the existence (above  $\kappa$ ) of  $\omega$  Woodin cardinals and a measurable cardinal above them. It is closely connected to the large cardinal strength of  $AD^{L(\mathbb{R})}$ .

Theorem 1 implies in particular that the example given above cannot be carried out with proper (as opposed to semiproper) forcing; the full theory of  $L(\mathbb{R})$ , with ordinal parameters, cannot be changed by proper forcing. It is a further immediate corollary of the Embedding Theorem that  $\text{HOD}^{L(\mathbb{R})}$  cannot be changed by proper forcing. Proper forcings also cannot “code” into  $L(\mathbb{R})$  a set of ordinals  $A \in V \setminus L(\mathbb{R})$ :

**Theorem 2 (Anti-coding Theorem)** *(Under large cardinal assumption  $A_\kappa$ ). Let  $P$  be a proper forcing notion of size  $\leq \kappa$ , and let  $G$  be  $P$ -generic/ $V$ . Suppose that  $A \subset \text{ON}$  is in  $V$ ; then  $A \in L(\mathbb{R}^V) \iff A \in L(\mathbb{R}^{V[G]})$ .*

As with the Embedding Theorem, the Anti-Coding Theorem cannot be extended much further. By a result of Woodin it fails for semiproper forcings (provably from large cardinals). Both the Anti-Coding Theorem and the Embedding Theorem do however extend to the class of reasonable forcings — a class slightly bigger than proper. The proofs in this paper apply to reasonable forcings.

Our Theorems are similar in flavor to results of Foreman and Magidor [FM95], who investigated the possibility of forcing to change the *definable continuum* — the supremum of

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<sup>1</sup>Ordinal parameters below  $\omega_2$  can, modulo  $\omega_1$ , be coded by reals. Thus, assuming large cardinals, forcing notions which preserve  $\omega_1$  cannot change the theory of  $L(\mathbb{R})$  with ordinal parameters below  $\omega_2$ .

all ordinals  $\gamma$  such that  $\gamma$  is the order type of some prewellorderings of reals in  $L(\mathbb{R})$ . (This ordinal is commonly denoted as  $\theta^{L(\mathbb{R})}$ ). In [FM95] it is shown that (granted large cardinals) reasonable forcings cannot change the definable continuum. This result can be obtained also from our Embedding Theorem, since clearly  $\theta^{L(\mathbb{R}^V)} = j(\theta^{L(\mathbb{R}^V)}) = \theta^{L(\mathbb{R}^{V[G]})}$  where  $G$  is  $P$ -generic/ $V$  for a reasonable forcing  $P$ , and  $j$  is the elementary embedding given by Theorem 1. [FM95] prove a more general result concerning prewellorderings which are homogeneously Suslin. Using additional work of Woodin's it is possible to derive the Embedding Theorem from their result. The proof of the Embedding Theorem which we include here is different, and its methods are needed later to obtain the Anti-Coding Theorem.

As with many results involving  $L(\mathbb{R})$  and large cardinals there are (at least) two alternative routes to proving our Theorems; one which uses stationary tower forcing, and another which uses iteration trees. The latter is presented in this paper while the former can be found in [NZ98]. It is interesting that even though iteration trees and stationary tower forcing are technically entirely different there are several similarities between the two approaches. Historically the Embedding and Anti-Coding Theorems were conjectured by the second author, who from a weakly compact Woodin cardinal proved the first for c.c.c forcings and the second for c.c.c. forcing as well as proper forcing notions contained in  $\omega_1$ . Both proofs used the techniques of stationary tower forcing. Those results were presented during the 1996 Set Theory meeting in Luminy, France. The first author subsequently used iteration trees to prove the full Theorems as they appear in the present paper, while the second author strengthened the stationary tower proofs to prove roughly the same results as they appear in [NZ98].

The structure of this paper is such that most of the use of large cardinal assumptions is exiled into two "black boxes" (Woodin's genericity iterations) which are quoted and then used. The proofs relating to these black boxes are due entirely to Hugh Woodin. Readers who are not experts on large cardinals may still be able to follow the proofs of the Embedding and Anti-Coding Theorems if they are willing to accept these black boxes. In Section 2 we present the proof of the Embedding Theorem, and in Section 3 the proof of the Anti-Coding Theorem. The proof in Section 3 uses the techniques of Section 2 as its backbone. The proofs of the black boxes are included in an Appendix to the e-print of this paper at <http://arXiv.org>.

## 2 The Embedding Theorem

We begin now the proof of the Embedding Theorem. Fix a proper forcing notion  $P$  and a generic  $G$ . Fix  $M$  which witnesses  $A_\kappa$ . To prove the Theorem we must construct the elementary embedding  $j: L(\mathbb{R}^V) \rightarrow L(\mathbb{R}^{V[G]})$ . The requirement that  $j \restriction \text{ON}$  be the identity essentially tells us what  $j$  is. We must have  $j(x) = x$  for any  $x \in \mathbb{R}^V$ , and since all elements of  $L(\mathbb{R})$  are definable from a real and some ordinals this fixes the map  $j$  completely. Any element of  $L(\mathbb{R}^V)$  definable in  $L(\mathbb{R}^V)$  from the real  $z$  and the ordinals  $\alpha_0, \dots, \alpha_k$  using the formula  $\phi$ , must be mapped to the element of  $L(\mathbb{R}^{V[G]})$  definable from  $z, \alpha_0, \dots, \alpha_k$  using the same formula  $\phi$  in  $L(\mathbb{R}^{V[G]})$ . All we must prove is that this gives a well-defined elementary embedding  $j$ , and this amounts to showing that

For any  $z \in \mathbb{R}^V$ , any  $\alpha_0, \dots, \alpha_k \in \text{ON}$ , and any formula  $\phi$ ,

$$\begin{aligned} L(\mathbb{R}^V) \models \phi[z, \alpha_0, \dots, \alpha_k] &\iff \\ L(\mathbb{R}^{V[G]}) \models \phi[z, \alpha_0, \dots, \alpha_k]. \end{aligned}$$

Fix  $z, \alpha_0, \dots, \alpha_k$ , and a formula  $\phi$ . We shall prove the above equivalence using a symmetric collapse. Given a model  $N$  and some ordinal  $\lambda$ , we consider the Lévy Collapse  $\text{col}(\omega, < \lambda)$  — the finite support product of the forcings  $\text{col}(\omega, \xi)$  for  $\xi < \lambda$ . Define the name  $\dot{\mathbb{R}}_{\text{symm}} = \bigcup_{\xi < \lambda} \mathbb{R}^{N[\dot{H} \restriction \text{col}(\omega, \xi)]}$ , where  $\dot{H}$  is a name for the generic object.  $\dot{\mathbb{R}}_{\text{symm}}$  are the reals in the symmetric collapse up to  $\lambda$ . Those were first investigated by Solovay who used a symmetric collapse to construct a model where all sets of reals are Lebesgue measurable. The important property of the collapse is its *homogeneity* — any statement about  $\dot{\mathbb{R}}_{\text{symm}}$  which involves only parameters from  $N$  is true in the generic extension iff it is forced by the *empty condition* (see [Jec78]).

Our strategy is to construct a model  $N$  and two different generics  $H_1$  and  $H_2$  such that

1.  $z \in N$ ;
2.  $H_1$  and  $H_2$  are both  $\text{col}(\omega, < \lambda)$ -generic/ $N$ ;
3.  $\dot{\mathbb{R}}_{\text{symm}}[H_1] = \mathbb{R}^V$ ; and
4.  $\dot{\mathbb{R}}_{\text{symm}}[H_2] = \mathbb{R}^{V[G]}$ .

This will immediately complete the proof, as

$$\begin{aligned} L(\mathbb{R}^V) \models \phi[z, \alpha_0, \dots, \alpha_k] &\iff \text{1} \quad N[H_1] \models “L(\dot{\mathbb{R}}_{\text{symm}}[H_1]) \models \phi[z, \alpha_0, \dots, \alpha_k]” \\ &\iff \text{2} \quad N \models “\Vdash^{\text{col}(\omega, < \lambda)} L(\dot{\mathbb{R}}_{\text{symm}}) \models \phi[z, \alpha_0, \dots, \alpha_k]” \\ &\iff \text{3} \quad N[H_2] \models “L(\dot{\mathbb{R}}_{\text{symm}}[H_2]) \models \phi[z, \alpha_0, \dots, \alpha_k]” \\ &\iff \text{4} \quad L(\mathbb{R}^{V[G]}) \models \phi[z, \alpha_0, \dots, \alpha_k]. \end{aligned}$$

The implications 1 and 4 follow from items (3) and (4) above. The implications 2 and 3 follow from the homogeneity of the forcing.

We construct  $N$  as an iterate of the model  $M$ , in  $\omega$  stages. Each stage will be carried out in  $V$  while the full construction will exist in  $V^{\text{col}(\omega, \mathbb{R})}$ . Our main tool is the following Theorem of Woodin’s (see [HMW] or <http://www.????.???>).

**Theorem (Woodin’s first genericity iteration)** *Let  $Q$  be an  $\omega_1+1$ -iterable inner model and let  $\tau < \eta$  be countable (in  $V$ ) ordinals such that  $Q \models “\eta$  is a Woodin cardinal”. Then there exists a forcing notion  $\mathbb{W}_{\tau, \eta}^Q \in Q$  of size  $\eta$ , such that for any real  $x$  it is possible to construct an iteration embedding  $j: Q \rightarrow \tilde{Q}$  with the property that*

- $x$  is  $j(\mathbb{W}_{\tau, \eta}^Q)$ -generic/ $\tilde{Q}$ ;
- $j(\eta)$  is countable in  $V$ , indeed  $j''(\omega_1^V) \subset \omega_1^V$ ; and
- $\text{crit}(j) > \tau$ .

Furthermore for any small forcing  $\mathbb{Q} \in V_\tau^Q$  there exists an  $\mathbb{Q}$  name for a forcing notion  $\dot{\mathbb{W}}_{\tau,\eta}^{Q,\mathbb{Q}}$  so that for any  $o$  which is  $\mathbb{Q}$ -generic/ $Q$ , there exists an iteration embedding  $j: Q \rightarrow \tilde{Q}$  satisfying the above except that now  $x$  is made  $j(\dot{\mathbb{W}}_{\tau,\eta}^{Q,\mathbb{Q}})[o]$ -generic/ $\tilde{Q}[o]$ .<sup>2</sup>

Woodin's genericity Theorem immediately tells us how to iterate  $M$  so as to satisfy condition (3) above. Fix some  $g: \omega \rightarrow \mathbb{R}^V$  which is  $\text{col}(\omega, \mathbb{R})$ -generic/ $V$ , and so enumerates all the reals of  $V$ . Our plan is to apply Woodin's Theorem using the  $2i$ -th Woodin cardinal of  $M$  to make  $g(i)$  generic over an iterate of  $M$ . (The reason we use only the even Woodin cardinals will become clear presently.) More precisely, working in  $M$  we let  $\{\delta_i\}_{i \in \omega}$  be an increasing sequence of Woodin cardinals with supremum  $\delta$ . Inductively define  $\mathbb{B}_i$  to be Woodin's forcing  $\dot{\mathbb{W}}_{\delta_{2i-1}, \delta_{2i}}^{M, \mathbb{B}_0 * \dots * \mathbb{B}_{i-1}}$  (defined in  $M$ ). Let  $\mathbb{B}$  be the finite support iteration of the forcings  $\mathbb{B}_i$ .

Inductively construct an iteration of  $M$ . Begin by letting  $M_0 = M$ , and construct models  $M_i$  and embeddings  $j_i: M_i \rightarrow M_{i+1}$  so that

- a.  $\langle g(0), \dots, g(i-1) \rangle$  is  $j_{0,2i}(\mathbb{B}_0 * \dots * \mathbb{B}_{i-1})$ -generic over  $M_{2i}$ , where  $j_{0,2i}$  is obtained through composition of the  $j_i$ 's.
- b.  $j_{2i}: M_{2i} \rightarrow M_{2i+1}$  is an embedding to make the real  $g(i)$  generic for the forcing notion  $j_{0,2i+1}(\mathbb{B}_i)[g(0), \dots, g(i-1)]$  over the model  $M_{2i+1}[g(0), \dots, g(i-1)]$ . (Such an embedding always exists by Woodin's first genericity iteration.) We take  $j_{2i}$  to be the identity whenever possible.
- c. For the time being, let  $j_{2i+1}: M_{2i+1} \rightarrow M_{2i+2}$  be the identity.

When iterating  $M$  we use the unique iteration strategy. Thus by an "iteration embedding of  $M$ " we mean only embeddings obtained through those iteration trees on  $M$  which choose the unique iterable branch at every limit stage. By our iterability assumption on  $M$  this guarantees that direct limits of iteration embeddings in  $V$  are well founded. Let  $M_\infty$  be the direct limit model of the  $M_i$ -s and let  $j_{i,\infty}: M_i \rightarrow M_\infty$  be the direct limit maps. Observe that  $M_\infty$  is well founded. This is not entirely trivial as the sequence  $j_i$  does not belong to  $V$ . However if this sequence gave rise to an ill founded direct limit one could use Schoenfield absoluteness to pull the existence of such a "bad" sequence back to  $V$ . Notice further that  $j_{2i,\infty}$  has critical point greater than  $j_{0,2i}(\delta_{2i-1})$  so that  $j_{0,\infty}(\mathbb{B}_0 * \dots * \mathbb{B}_{i-1}) = j_{0,2i}(\mathbb{B}_0 * \dots * \mathbb{B}_{i-1})$  and  $j_{0,\infty}(\delta_{2i-1}) = j_{0,2i}(\delta_{2i-1})$ . In particular,  $j_{0,\infty}(\delta_{2i-1})$  is countable (in  $V$ ) and so  $j_{0,\infty}(\delta) \leq \omega_1^V$ .

It is clear from the construction that  $\langle g(0), \dots, g(i-1) \rangle$  is  $j_{0,\infty}(\mathbb{B}_0 * \dots * \mathbb{B}_{i-1})$ -generic/ $M_\infty$  for all  $i$ . Using the fact that  $g$  is  $\text{col}(\omega, \mathbb{R}^V)$ -generic/ $V$  one can verify further that  $\langle g(i) \mid i < \omega \rangle$  is  $j_{0,\infty}(\mathbb{B})$ -generic over  $M_\infty$ . Specifically, fix any dense set  $D$  in  $j_{0,\infty}(\mathbb{B})$  and assume (\*) that the filter given by  $\langle g(i) \mid i < \omega \rangle$  does not intersect  $D$ . Fix some  $n$  large enough so that  $\langle g(0), \dots, g(n-1) \rangle$  forces (\*) in  $\text{col}(\omega, \mathbb{R})$ , and such that  $D = j_{2n,\infty}(\bar{D})$  for some  $\bar{D} \in M_{2n}$ . Now by condition (a),  $\langle g(0), \dots, g(n-1) \rangle$  is  $j_{0,2n}(\mathbb{B}_0 * \dots * \mathbb{B}_{n-1})$ -generic/ $M_{2n}$ . Working in  $M_{2n}[g(0), \dots, g(n-1)]$  we can therefore find a condition  $b = \langle b_0, \dots, b_{k-1} \rangle \in \bar{D}$

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<sup>2</sup> This does not follow from the previous part of the Theorem, since it gives a  $j$  which is an iteration of  $Q$ , and this is more restrictive than being an iteration of  $Q[o]$ .

(with  $k \geq n$ ) such that  $\langle \dot{b}_0, \dots, \dot{b}_{n-1} \rangle$  belongs to the  $j_{0,2n}(\mathbb{B}_0 * \dots * \mathbb{B}_{n-1})$ -generic given by  $\langle g(0), \dots, g(n-1) \rangle$ . Next let us force over  $M_{2n}$  with  $j_{0,2n}(\mathbb{B}_n * \dots * \mathbb{B}_{k-1})[g(0), \dots, g(n-1)]$  below the condition  $\langle \dot{b}_n, \dots, \dot{b}_{k-1} \rangle[g(0), \dots, g(n-1)]$ , and obtain reals  $y_n, \dots, y_{k-1}$  such that

G1.  $\langle g(0), \dots, g(n-1), y_n, \dots, y_{k-1} \rangle$  is  $j_{0,2n}(\dot{\mathbb{B}}_0 * \dots * \dot{\mathbb{B}}_{k-1})$ -generic/ $M_{2n}$ , and

G2. this generic contains the condition  $b \in \bar{D}$ .

Such reals can be found in  $V$  since the level of  $M_{2n}$  involved is countable in  $V$  (see conditions (i,ii) below). Consider finally the condition  $\langle g(0), \dots, g(n-1), y_n, \dots, y_{k-1} \rangle$  in the forcing  $\text{col}(\omega, \mathbb{R})$ . This condition forces our construction to produce a model  $M_{2k}$  which is equal to  $M_{2n}$ , and an embedding  $j_{2n,2k}$  which equals the identity (note our use here of the requirement in (b) that  $j_{2i}$  be the identity whenever possible). From this together with (G1,G2) it follows easily that  $(*)$  is forced to *fail*, but this is a contradiction.

Observe next that the forcing  $\mathbb{B}$  can be replaced by a symmetric collapse. In other words it is possible to find  $H \in M_\infty[g(i) \mid i < \omega]$  which is  $\text{col}(\omega, < j_{0,\infty}(\delta))$ -generic/ $M_\infty$  and so that  $\mathbb{R}_{\text{symm}}^{M_\infty}[H] = \{g(i) \mid i < \omega\}$ . In fact it is well known that in general (for  $\delta$  a strong limit cardinal) whenever  $\mathbb{A}$  is a direct limit of a regular chain of forcings  $\mathbb{A}_i$ , each of size  $< \delta$ , such that each cardinal below  $\delta$  is collapsed to  $\omega$  by some  $\mathbb{A}_i$ , then  $\mathbb{A}$  is isomorphic to  $\text{col}(\omega, < \delta)$  in such a way that the symmetric reals are exactly those added by the forcings  $\mathbb{A}_i$ . In our case the reals added by the forcings  $\mathbb{B}_0 * \dots * \mathbb{B}_i$  are all in  $V$ , and eventually all reals of  $V$  are added. Thus we finally have  $\mathbb{R}^V = \mathbb{R}_{\text{symm}}^{M_\infty}[H]$ .

The argument so far is not new. It was first presented by Steel who used it in [Ste93] to derive several absoluteness results for  $L(\mathbb{R})$ , among them the generic absoluteness of the theory of  $L(\mathbb{R})$  with real—but not ordinal—parameters. For our purposes however this argument is not sufficient. We have made  $\mathbb{R}^V$  the set of reals in the symmetric collapse of an iterate of  $M$ , but we must simultaneously make  $\mathbb{R}^{V[G]}$  the set of reals in a different symmetric collapse of the same iterate. For this reason exactly we left ourselves some space during the construction, in the form of the embeddings  $j_{2i+1,2i+2}$  and the Woodin cardinals  $\delta_{2i+1}$ . Let  $\dot{\mathbb{C}}_i$  be Woodin's forcing  $\dot{\mathbb{W}}_{\delta_{2i}, \delta_{2i+1}}^{M, \dot{\mathbb{C}}_0 * \dots * \dot{\mathbb{C}}_{i-1}}$  defined in  $M$ , and  $\mathbb{C}$  their finite support iteration. We will use those to make the reals of  $V[G]$  generic, as we made the reals of  $V$  generic. We must however take care not to spoil the part of the construction we have completed—we want to define  $j_{2i+1,2i+2}$  in a way that still allows us to argue that  $\langle g(i) \mid i < \omega \rangle$  is generic for  $j_{0,\infty}(\mathbb{B})$ . For that argument to work we needed to know that the reals  $y_n, \dots, y_{k-1}$  could be chosen in  $V$ , and this followed from

i.  $V_{j_{0,\infty}(\delta_i)}^{M_\infty}$  is an element of  $V$  for all  $i$ ; and

ii.  $j_{0,\infty}(\delta_i)$  is countable in  $V$ , for all  $i$ .

Either one of (i),(ii) can easily be maintained using Woodin's first and second (see below) genericity iterations. The difficulty is in maintaining both conditions simultaneously, and it is here that we must make use of our assumption that  $P$  is proper.

**Lemma 3** (Assuming  $G$  is  $P$ -generic/ $V$  for some proper  $P$ .) Let  $Q = L(V_\delta^Q)$  be uniquely iterable, in  $V$ , for trees of size  $\kappa^+ + 1$ . Assume  $\delta$  is countable in  $V$ , let  $\tau < \eta < \delta$  be ordinals

such that  $Q \models \text{“}\eta \text{ is a Woodin cardinal”}$ , and consider Woodin’s forcing  $\mathbb{W} = \mathbb{W}_{\tau, \eta}^Q$ . **Then** for any real  $x \in V[G]$  it is possible to construct an iteration embedding  $j: Q \rightarrow \tilde{Q}$  in  $V$  with the property that

- $x$  is  $j(\mathbb{W})$ -generic/ $\tilde{Q}$ ;
- $j(\eta)$  is countable in  $V$ , indeed  $j''(\omega_1^V) \subset \omega_1^V$ ; and
- $\text{crit}(j) > \tau$ .

Furthermore for any small forcing  $\mathbb{O} \in V_\tau^Q$ , if we let  $\dot{\mathbb{W}} = \dot{\mathbb{W}}_{\tau, \eta}^{Q, \mathbb{O}}$  then for any  $o \in V[G]$  which is  $\mathbb{O}$ -generic/ $Q$  it is possible to construct an iteration embedding  $j: Q \rightarrow \tilde{Q}$  satisfying the above except that now  $x$  is made  $j(\dot{\mathbb{W}})[o]$ -generic over  $\tilde{Q}[o]$ .

It is worthwhile emphasizing the difference between Woodin’s Theorem and Lemma 3. In Lemma 3 we allow  $x \in V[G]$  (and also  $o \in V[G]$  for the second part), and still obtain an iteration embedding  $j$  in  $V$ . Fix  $g: \omega \rightarrow \mathbb{R}^V$  and  $h: \omega \rightarrow \mathbb{R}^{V[G]}$  so that the pair  $g, h$  is  $\text{col}(\omega, \mathbb{R}^V) \times \text{col}(\omega, \mathbb{R}^{V[G]})$ -generic/ $V[G]$ . Granted the Lemma we may repeat our construction replacing condition (c) with

- c'.  $j_{2i+1}: M_{2i+1} \rightarrow M_{2i+2}$  is an embedding to make the real  $h(i)$  generic for the forcing  $j_{0, 2i+2}(\dot{\mathbb{C}}_i)[h(0), \dots, h(i-1)]$  over the model  $M_{2i+2}[h(0), \dots, h(i-1)]$ . We take  $j_{2i+1}$  to be the identity if possible. Otherwise we take the embedding given by Lemma 3.

This modified construction produces  $M_\infty$  and  $j_{i, \infty}$  satisfying

1. For all  $n < \omega$   $\langle g(0), \dots, g(n-1) \rangle$  is  $j_{0, \infty}(\dot{\mathbb{B}}_0 * \dots * \dot{\mathbb{B}}_{n-1})$ -generic/ $M_\infty$ ;
2. For all  $n < \omega$   $\langle h(0), \dots, h(n-1) \rangle$  is  $j_{0, \infty}(\dot{\mathbb{C}}_0 * \dots * \dot{\mathbb{C}}_{n-1})$ -generic/ $M_\infty$ ; and
3. For  $\xi < j_{0, \infty}(\delta)$ ,  $V_\xi^{M_\infty}$  belongs to  $V$  and is countable in  $V$ .

Condition (3) and the genericity of  $g, h$  allow us as before to argue that in fact  $\langle g(i) \mid i < \omega \rangle$  is  $j_{0, \infty}(\mathbb{B})$ -generic/ $M_\infty$ ; and  $\langle h(i) \mid i < \omega \rangle$  is  $j_{0, \infty}(\mathbb{C})$ -generic/ $M_\infty$ .

As before we can now convert the forcings  $\mathbb{B}$  and  $\mathbb{C}$  into symmetric collapses — finding  $H_1$  and  $H_2$  which are  $\text{col}(\omega, < j_{0, \infty}(\delta))$ -generic/ $M_\infty$  so that  $\dot{\mathbb{R}}_{\text{symm}}^{M_\infty}[H_1] = \{g(i) \mid i < \omega\}$  and  $\dot{\mathbb{R}}_{\text{symm}}^{M_\infty}[H_2] = \{h(i) \mid i < \omega\}$ . Letting  $N = M_\infty$  this completes the proof of Theorem 1, at least if  $z$  belongs to  $M$  — but if not, before the beginning of the construction simply iterate  $M$  to make  $z$  generic, and then continue to realize  $\mathbb{R}^V$  and  $\mathbb{R}^{V[G]}$  as the reals of a symmetric collapse over  $N[z]$ .

It remains therefore only to prove Lemma 3. We use the following:

**Theorem (Woodin’s second genericity iteration)** *Let  $Q$  be a  $\kappa^+ + 1$ -iterable inner model, let  $\tau < \eta < \kappa^+$  be ordinals such that  $Q \models \text{“}\eta \text{ is a Woodin cardinal”}$ , and let  $\mathbb{A} \in V$  be any forcing notion of size  $\leq \kappa$ . Let  $\mathbb{W} = \mathbb{W}_{\tau, \eta}^Q$  be Woodin’s forcing of the first genericity iteration, defined in  $Q$  from  $\tau$  and  $\eta$ . **Then** for any  $\dot{x}$  which is a name for a real in  $V^\mathbb{A}$ , it is possible to construct an iteration embedding  $j: Q \rightarrow \tilde{Q}$  (in  $V$ ) with the property that*

- For any  $F$  which is  $\mathbb{A}$ -generic/ $V$ , the real  $\dot{x}[F]$  is  $j(\mathbb{W})$ -generic/ $\tilde{Q}$ ;
- $j(\eta) < (\kappa^+)^V$ , indeed  $j''(\kappa^+) \subset \kappa^+$ ; and
- $\text{crit}(j) > \tau$ .

Furthermore For any small forcing  $\mathbb{O} \in V_\tau^Q$ , if we let  $\mathbb{W} = \dot{\mathbb{W}}_{\tau,\eta}^{M,\mathbb{O}}$  then for any  $\dot{o}$  which is an  $\mathbb{A}$  name for an  $\mathbb{O}$ -generic filter/ $Q$ , it is possible to construct an iteration embedding  $j: Q \rightarrow \tilde{Q}$  (in  $V$ ) satisfying the above except that now  $\dot{x}[F]$  is made generic over  $\tilde{Q}[\dot{o}[F]]$  (for all  $F$  which are  $\mathbb{A}$ -generic/ $V$ ).

Using Woodin's second genericity iteration let us prove Lemma 3. Let  $j^*: Q \rightarrow Q^*$  be the embedding given by Woodin's second genericity iteration applied with a name  $\dot{x}$  for the real  $x \in V[G]$ . Then  $j^* \in V$ , but  $j^*(\eta)$  need not be countable. To overcome this: Fix an elementary submodel  $Y$  of  $V_\lambda$  for some sufficiently large  $\lambda$  so that  $P, \dot{x}, Q, j^*, Q^* \in Y$ <sup>3</sup>;  $Y$  belongs to  $V$  and is countable in  $V$ ;  $G \cap Y$  is  $P$ -generic/ $Y$ ; and  $Y[G \cap Y] \prec V_\lambda[G]$ . The existence of  $Y$  follows from the properness of  $P$ . In fact it is enough (by the very definition) to assume that  $P$  is reasonable. Let  $\bar{Y}$  be the transitive collapse of  $Y$  and  $\pi: \bar{Y} \rightarrow Y$  the inverse collapse embedding. Let  $\bar{G} = \pi^{-1}''G$  and  $\bar{x}, j, \tilde{Q} = \pi^{-1}(\dot{x}, j^*.Q^*)$ . Let  $\bar{x} = \bar{x}[\bar{G}]$ . Notice that  $Q$  is not moved by  $\pi^{-1}$ , so we have  $j: Q \rightarrow \tilde{Q}$ .  $\pi$  induces an embedding from  $\bar{Y}[\bar{G}]$  onto  $Y[G \cap Y]$  which we also call  $\pi$ . Thus  $\pi: \bar{Y}[\bar{G}] \rightarrow V_\lambda[G]$  is elementary.

By the elementarity of  $\pi$ ,  $\bar{x}$  is  $j(\mathbb{W}_{\tau,\eta}^Q)$ -generic/ $\tilde{Q}$ . Of course  $\bar{x}$  is a real and is not moved by  $\pi$ , so  $\bar{x} = \pi(\bar{x}) = x$ . Thus the embedding  $j$  makes  $x$  generic for Woodin's forcing. As  $j \in \bar{Y}$  it is clear that  $j(\eta)$  is countable in  $V$ .

The reader can now easily check the remaining requirements of Lemma 3. Let us here only verify that  $j$  is an iteration embedding. This is not obvious — by elementarity  $\bar{Y} \models$  “ $j$  is an iteration embedding”, but this does not mean  $j$  is an iteration embedding in  $V$ . Let  $\mathcal{T}$  be the iteration tree giving rise to  $j$ . We must show that the branches  $\mathcal{T}$  chooses are according to the iteration strategy for  $Q$  which we have in  $V$ . But  $Q$  is *uniquely* iterable, so this strategy chooses at every limit stage the unique branch with iterable direct limit. Thus it is sufficient to show that every model  $Q_\xi$  on the tree  $\mathcal{T}$  is iterable (in  $V$ ). Remember that  $\pi$  maps  $Q_\xi$  into a model  $Q_{\xi^*}^*$  on the tree  $\mathcal{T}^*$  which gives rise to  $j^*$ .  $Q_{\xi^*}^*$  is iterable and by [MS94] every model which embeds into an iterable model is iterable. Thus  $Q_\xi$  is iterable and we are done.

The second part of Lemma 3 is proved in a similar fashion. Note that since  $Q$  is countable and  $\dot{o}, \mathbb{O} \in Q$ , both  $\dot{o}$  and  $\mathbb{O}$  are automatically in  $Y$  and  $\pi(\dot{o}) = \dot{o}$ . Thus both  $\dot{x}[G]$  and  $\dot{o}[G]$  are not moved by  $\pi: \bar{Y}[\bar{G}] \rightarrow V_\lambda[G]$ . We take  $j^*$  to be the iteration from the second part of Woodin's second genericity Theorem, and immediately by the elementarity of  $\pi$  can conclude that  $\dot{x}[G]$  is generic over  $\tilde{Q}[\dot{o}[G]]$ . □(Lemma 3, Theorem 1)

### 3 The Anti-Coding Theorem

Next let us prove Theorem 2. Fix a set  $A \subset \text{ON}$  in  $V$ . We must show that  $A \in L(\mathbb{R}^V)$  iff  $A \in L(\mathbb{R}^{V[G]})$ . Now the implication from left to right follows immediately from the

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<sup>3</sup> $Q$  is a class model of course, but it is coded by a real, and we can throw this real into  $Y$ .



Embedding Theorem. Assume then that  $A \in L(\mathbb{R}^{V[G]})$ . We must show  $A \in L(\mathbb{R}^V)$ . As all sets in  $L(\mathbb{R})$  are definable from a real and some ordinals, we may fix a name  $\dot{x}$ , ordinals  $\vec{\alpha}$  and a formula  $\phi$ , so that

$$\dot{A}[G] = A = \{ \gamma \mid L(\mathbb{R}^{V[G]}) \models \phi[\vec{\alpha}, \dot{x}[G], \gamma] \}.$$

Without loss of generality we may assume that this is forced by the empty condition in  $P$ .

It is convenient to replace  $A_\kappa$  with the large cardinal assumption  $B_\kappa$  stated below. It can be seen (using Woodin's second genericity iteration and some fine structure) that  $B_\kappa$  follows from  $A_\kappa$ .

( $B_\kappa$ ) For any  $K \subset \kappa$  there exists a class model  $M$  such that

- $M = L(V_\delta^M)$ , for some  $\kappa < \delta < (\kappa^+)^V$ ;
- $V_\kappa \subset M$ , and  $K \in M$ ;
- $M \models \text{"}\delta \text{ is the supremum of } \omega \text{ Woodin cardinals;"} \text{ and}$
- $M$  is uniquely iterable above  $\kappa$  for trees of length  $\leq (\kappa^+)^V$  (i.e., the good player wins the iteration game when the bad player is restricted to playing extenders with critical points above  $\kappa$ ).

As the forcing  $P$  has size  $\kappa$  we may take it to be a subset of  $\kappa$ , and so can fix a model  $M$  satisfying the conditions of assumption  $B_\kappa$  with  $P, \dot{x} \in M$ . Notice that from  $B_\kappa$  it follows that every subset of  $\kappa$  has a sharp, and so  $V_\delta^M$  has a sharp.

We now pass to work in a countable elementary submodel  $Y \prec V_\lambda$  (for  $\lambda$  sufficiently large) which belongs to  $V$ , and contains all relevant objects (including  $V_\delta^M$  and its sharp). Let  $\bar{Y}$  be the transitive collapse of  $Y$ , and  $\bar{M}$  the image under the collapse map of  $M$ .<sup>4</sup> Let  $\pi: \bar{Y} \rightarrow Y$  be the inverse collapse embedding. Let  $\bar{\dot{x}}, \bar{P}$ , and  $\bar{G}$  be the collapse of  $\dot{x}$ ,  $P$ , and  $G \cap Y$ . Then  $\bar{\dot{x}}[\bar{G}] = x$ , and by properness (reasonability) of  $P$  we may assume that  $\bar{G}$  is  $\bar{P}$ -generic/ $\bar{Y}$ . We will attempt to replace the real  $\bar{\dot{x}}[\bar{G}]$  in the definition of  $A$  with a real  $\bar{\dot{x}}[h]$  for some  $h \in V$  which is  $\bar{P}$ -generic/ $\bar{Y}$ . The fact that  $h \in V$  will then imply that  $A \in L(\mathbb{R}^V)$ . It is simple to find  $h \in V$  which is  $\bar{P}$ -generic/ $\bar{Y}$  (since  $\bar{Y}$  is countable). The difficulty of course is to do this in such a way that  $\bar{\dot{x}}[\bar{G}]$  and  $\bar{\dot{x}}[h]$  still define the same set of ordinals.

Let us find  $h \in V$  which is  $\bar{P}$ -generic/ $\bar{Y}$  with the property that for any  $\bar{p} \in h$ , there exists a condition  $q \leq \pi(\bar{p})$  which is  $(Y, P)$ -generic. If  $P$  is proper this can be done trivially (perhaps at the price of modifying  $Y$ ). If  $P$  is only known to be reasonable this is a bit less trivial. Fix in this case some  $q_0 \in G$  which is  $(Y, P)$ -generic. In  $V[G]$  there exists an  $h$  which is  $\bar{P}$ -generic/ $\bar{Y}$  such that all conditions in  $\pi''(h)$  are compatible with  $q_0$  (e.g. take  $h = \bar{G}$ ). By absoluteness then such  $h$  exists in  $V$ , and it is easy to see that any such  $h$  satisfies our requirement above.

Through our choice of  $h$  we may, for any condition  $\bar{p} \in h$ , fix in some external generic extension of  $V$  a filter  $G^{\bar{p}}$  which is  $P$ -generic/ $V$ ; contains the condition  $\pi(\bar{p})$ ; and such that  $\bar{G}^{\bar{p}} = \pi^{-1}''(G^{\bar{p}} \cap Y)$  is  $\bar{P}$ -generic over  $\bar{Y}$  (and hence also over  $\bar{M}$ <sup>5</sup>). By dovetailing together constructions of the sort used in Section 2 iterate  $\bar{M}$  to a model  $N$  so that

<sup>4</sup>Again,  $M$  is a class model. What we mean is that  $\bar{M} = L(V_\delta^{\bar{M}})$  where  $V_\delta^{\bar{M}}$  is the collapse of  $V_\delta^M$ .

<sup>5</sup>We are using here the existence of  $V_\delta^{\bar{M}}$  inside  $\bar{Y}$  to see that all subsets of  $\bar{P}$  in  $\bar{M}$  belong to  $\bar{Y}$ .

- a. For each  $\bar{p} \in h$  the reals of  $V[G^{\bar{p}}]$  can be realized as the symmetric collapse over  $N[\bar{G}^{\bar{p}}]$ ; and
- b. The reals of  $V$  can be realized as the symmetric collapse over  $N[h]$ .

Let  $j: \bar{M} \rightarrow N$  be the iteration embedding, which we construct to have critical point above  $\bar{\kappa}$ , so that  $j(\bar{P}) = \bar{P}$ ,  $j(\bar{x}) = \bar{x}$  etc. As in Section 2  $j$  is a composition of  $\omega$  maps, each of which is in  $V$ , and  $j$  itself exists only in some external model. Let us denote  $j(\bar{\delta})$  by  $\tilde{\delta}$ .<sup>6</sup> “The symmetric collapse” in (a,b) above refers to the collapse up to  $\tilde{\delta}$ .

**Claim 4** *Working in  $N[h]$  let  $y = \bar{x}[h]$  and consider the forcing  $\text{col}(\omega, < \tilde{\delta})$ . We claim that for any ordinal  $\gamma$  the following are equivalent:*

- 1.  $\gamma \in A$
- 2. In the forcing  $\text{col}(\omega, < \tilde{\delta})$  over  $N[h]$ , it is forced that “ $L(\dot{\mathbb{R}}_{\text{symm}}) \models \phi[\vec{\alpha}, y, \gamma]$ ”.

Otherwise we may fix some ordinal  $\gamma$ , and a condition  $\bar{p} \in h$ , such that  $\gamma \notin A$  say, and nonetheless  $\bar{p}$  forces in  $\bar{P}$  that  $L(\dot{\mathbb{R}}_{\text{symm}}) \models \phi[\vec{\alpha}, \bar{x}, \gamma]$  holds in the symmetric collapse. (Alternatively  $\gamma \in A$  and  $\bar{p}$  forces  $L(\dot{\mathbb{R}}_{\text{symm}}) \not\models \phi[\vec{\alpha}, y, \gamma]$ , but the proof in this case is similar.) Since  $\bar{p}$  is an element of  $\bar{G}^{\bar{p}}$  it follows that over  $N[\bar{G}^{\bar{p}}]$  it is forced in  $\text{col}(\omega, < \tilde{\delta})$  that  $L(\dot{\mathbb{R}}_{\text{symm}}) \not\models \phi[\vec{\alpha}, \bar{x}[\bar{G}^{\bar{p}}], \gamma]$ . But now by (a) we may fix  $H_{\bar{p}}$  which is  $\text{col}(\omega, < \tilde{\delta})$ -generic/ $N[\bar{G}^{\bar{p}}]$  such that  $\dot{\mathbb{R}}_{\text{symm}}[H_{\bar{p}}] = \mathbb{R}^{V[G^{\bar{p}}]}$ . As furthermore  $\bar{x}[\bar{G}^{\bar{p}}] = \bar{x}[G^{\bar{p}}]$  it follows that  $L(\mathbb{R}^{V[G^{\bar{p}}]}) \not\models \phi[\vec{\alpha}, \bar{x}[G^{\bar{p}}], \gamma]$ . But this implies  $\gamma \notin A$ , a contradiction.  $\square$ (Claim 4)

By (b) we may fix  $H$ , a filter which is  $\text{col}(\omega, < \tilde{\delta})$ -generic over  $N[h]$  so that  $\dot{\mathbb{R}}_{\text{symm}}[H] = \mathbb{R}^V$ . By Claim 4  $\gamma \in A$  iff  $L(\dot{R}_{\text{symm}}[H]) \models \phi[\vec{\alpha}, y, \gamma]$ , i.e.,  $L(\mathbb{R}^V) \models \phi[\vec{\alpha}, y, \gamma]$ . Thus,

$$A = \{ \gamma \mid L(\mathbb{R}^V) \models \phi[\vec{\alpha}, y, \gamma] \} \in L(\mathbb{R}^V)$$

completing the proof of the Anti-Coding Theorem.  $\square$ (Theorem 2)

## A Black Boxes

We include here a proof of Woodin’s genericity Theorems. The results in this Appendix are due to Hugh Woodin (circa 1987, to be published in [HMW]). The reader may easily verify that Woodin’s first genericity iteration is an immediate corollary of the second (taking  $\mathbb{A}$  to be the trivial forcing for adding nothing and  $\kappa = \omega$ ), and so we prove here only the second. For the rest of this section  $\eta$  is assumed to be a Woodin cardinal in  $Q$ .

Consider the algebra  $\mathcal{L}_\eta$  of all transfinite formulae formed by starting with “ $n \in \tilde{x}$ ” (for  $n \in \omega$ ) and closing under negation and wellordered disjunctions of length  $< \eta$ . The forcing  $\mathbb{W}_{\tau, \eta}^Q$  is similar to the Lindenbaum algebra on  $\mathcal{L}_\eta$ , but rather than simply setting  $\phi \leq \psi \iff \vdash \phi \rightarrow \psi$  Woodin introduces a set of *axioms*  $\mathcal{A} \subset \mathcal{L}_\eta$  and then defines:

$$\phi \approx \psi \text{ iff } \mathcal{A} \vdash \phi \leftrightarrow \psi; \text{ and } [\phi] \leq [\psi] \text{ iff } \mathcal{A} \vdash \phi \rightarrow \psi.$$

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<sup>6</sup>This is easily seen to be equal to  $\omega_1^V$ .

$\mathbb{W}_{\tau,\eta}^Q$  is defined to be the forcing notion consisting of equivalence classes  $[\phi]$  for  $\phi \in \mathcal{L}_\eta$ , ordered by  $\leq$  as above.

Before writing down the set of axioms  $\mathcal{A}$  note that with this definition, if  $\phi$  is any formula such that  $[\phi] = 0$  and  $x$  any real such that  $x \models \phi$ , then there must exist an axiom  $a \in \mathcal{A}$  such that  $x \models \neg a$ . Thus any real satisfying the axioms cannot satisfy the 0 condition.

The set  $\mathcal{A}$  is defined as follows: For any  $\lambda$  and  $\rho$  satisfying  $\tau < \lambda < \rho < \eta$ , any  $\rho$ -strong extender  $E \in V_\eta^Q$  with  $\text{crit}(E) = \lambda$ , and any sequence  $\vec{\phi} = \{\phi_\xi\}_{\xi < \lambda}$  of formulae in  $\mathcal{L}_\eta \cap V_\lambda^Q$ , let  $i_E: Q \rightarrow \text{Ult}(Q, E)$  be the ultrapower embedding of  $Q$ , and let  $\nu$  be least such that  $i_E(\vec{\phi})_\nu$  is not in  $V_\rho^Q$ . (Notice that  $\nu \geq \lambda$ , and certainly a strict inequality is possible.) The following formula is taken to be an axiom:

$$a_{\lambda,\rho,E,\vec{\phi}} = "[\bigvee_{\xi < \nu} i_E(\vec{\phi})_\xi] \rightarrow [\bigvee_{\xi < \lambda} \phi_\xi]."$$

(This is a formula in  $\mathcal{L}_\eta$ , and in fact one which is an element of  $V_{\rho+1}^Q$ .) We denote  $\nu$  by  $\nu_{\lambda,\rho,E,\vec{\phi}}$ . It is worthwhile observing that  $i_E(\vec{\phi})_\xi = \phi_\xi$  for  $\xi < \lambda$ , so that the disjunction  $\bigvee_{\xi < \nu} i_E(\vec{\phi})_\xi$  is always weaker than (or equal to) the disjunction  $\bigvee_{\xi < \lambda} \phi_\xi$ .

Woodin then proves the following Claim

**Claim** *In  $Q$ , the forcing  $\mathbb{W}_{\tau,\eta}^Q$  is  $\eta$ -c.c.*

**Proof.** Assume for contradiction that the Claim fails and fix an anti-chain  $\{[\psi_\xi]\}_{\xi < \eta}$  witnessing this. Let  $f: \eta \rightarrow \eta$  be the function defined by setting  $f(\xi)$  to be least  $\alpha$  such that  $\psi_\xi \in V_\alpha^Q$ . Since  $\eta$  is a Woodin cardinal we can now find  $\lambda < \rho$  between  $\tau$  and  $\eta$  and an extender  $E \in V_\eta^Q$  such that

1.  $\text{crit}(E) = \lambda$ ;
2.  $E$  is  $\rho$  strong, and indeed even  $\rho$  strong wrt  $\{\psi_\xi \mid \xi < \eta\}$ ; and
3.  $\rho > i_E(f)(\lambda)$ .

Let  $\vec{\phi} = \vec{\psi} \upharpoonright \lambda$ , and consider the axiom  $a_{\lambda,\rho,E,\vec{\phi}}$ . By condition (3)  $\nu_{\lambda,\rho,E,\vec{\phi}} \geq \lambda + 1$  so  $a_{\lambda,\rho,E,\vec{\phi}}$  clearly proves  $i_E(\vec{\phi})_\lambda \rightarrow \bigvee_{\xi < \lambda} \phi_\xi$ . But by condition (2)  $i_E(\vec{\phi})_\lambda = \psi_\lambda$  and so  $\mathcal{A} \vdash "\psi_\lambda \rightarrow \bigvee_{\xi < \lambda} \psi_\xi"$ . Thus  $[\psi_\lambda] \leq \bigvee_{\xi < \lambda} [\psi_\xi]$  — a contradiction since  $\{[\psi_\xi]\}_{\xi < \eta}$  is an anti-chain.  $\square$

**Lemma (Woodin)** *Let  $x$  be any real and assume that  $x \models a$  for all  $a \in \mathcal{A}$ . Then  $x$  generates a  $\mathbb{W}_{\tau,\eta}^Q$ -generic filter  $W_x$ , such that  $x \in Q[W_x]$ .*

**Proof.** Define  $W_x = \{[\phi] \in \mathbb{W}_{\tau,\eta}^Q \mid x \models \phi\}$ . This is well defined since  $x \models \mathcal{A}$ . To see that  $W_x$  is a generic filter: Let  $\{[\psi_\xi]\}_{\xi < \beta}$  be a maximal anti-chain in  $\mathbb{W}_{\tau,\eta}^Q$  and assume for contradiction that  $[\psi_\xi] \notin W_x$  for all  $\xi < \beta$ . Note that  $\beta < \eta$  by the previous Claim, so  $\varphi = \bigvee_{\xi < \beta} \neg \psi_\xi$  is a formula in  $\mathcal{L}_\eta$ .  $[\varphi]$  is therefore a condition, and  $[\varphi] = 0$  since  $\{[\psi_\xi]\}_{\xi < \beta}$  is a maximal anti-chain. But  $x \models \varphi$  and this is a contradiction since  $x \models \mathcal{A}$ .

Finally to see  $x \in Q[W_x]$ , note that  $x = \{n \mid ["n \in \tilde{x}"] \in W_x\}$ .  $\square$

At last we are in a position to prove Woodin's second genericity Theorem. Fix a forcing notion  $\mathbb{A}$  of size  $\kappa$  and let  $\dot{x}$  be a name for a real in  $V^{\mathbb{A}}$ . By the previous Lemma, the real  $\dot{x}$  is generic over  $Q$  unless it contradicts some of the axioms in  $\mathcal{A}$ . The reader can easily verify that if a real  $z$  contradicts some axiom  $a_{\lambda, \rho, E, \vec{\phi}}$ , then  $z$  does *not* contradict the image axiom  $i_E(a_{\lambda, \rho, E, \vec{\phi}})$ , where  $i_E: Q \rightarrow \text{Ult}(Q, E)$  is the ultrapower map. Thus forming the ultrapower by  $E$  "removes" the obstruction caused by the axiom  $a_{\lambda, \rho, E, \vec{\phi}}$ . The second genericity iteration is proved by forming an iteration tree, hitting at every stage the first extender  $E$  which defines an axiom that  $\dot{x}$  does not satisfy. A comparison type argument is then used to show that this iteration terminates. The key to this comparison type argument is the fact that once an obstructing axiom has been removed its image will never again become an obstructing axiom. Thus with each step of the construction we come closer to having no obstructing axioms at all. This argument requires an iteration *tree*; if instead we attempt to use linear iterations then each step may undo previous steps, and the image of an axiom that was handled previously may become obstructing again.

Let us begin the construction. We construct a normal iteration tree  $\mathcal{T} = \langle E_\alpha, \rho_\alpha \mid \alpha < \beta \rangle$  with models  $Q_\alpha$  and tree structure  $T$ . The construction is inductive. At limit  $\alpha$  we use our iteration strategy for  $Q$  to pick a cofinal branch of the tree  $\mathcal{T} \restriction \alpha$ , and set  $Q_\alpha$  to be the direct limit of the models along this branch. At successor stages  $\alpha + 1$  we must specify  $E_\alpha$  and  $\rho_\alpha$  (the tree structure is then determined by finding the least  $\alpha'$  such that  $\rho_{\alpha'} > \text{crit}(E)$  and setting  $\alpha' T \alpha + 1$ ). We shall use only extenders with critical point above  $\tau$ .

At successor stages we distinguish between two cases.

**Case 1:** If  $\dot{x}[F]$  is  $j_{0, \alpha}(\mathbb{W}_{\tau, \eta}^Q)$ -generic/ $Q_\alpha$  (for all  $\mathbb{A}$ -generic/ $V$  filters  $F$ ) then we let  $\beta = \alpha$ ,  $j = j_{0, \alpha}$ , and we are done proving the Theorem.

**Case 2:** Otherwise, working in  $V^{\mathbb{A}}$  we apply the previous Lemma to  $Q_\alpha$  and  $\mathbb{W}_{\tau, j_{0, \alpha}(\eta)}^{Q_\alpha} = j_{0, \alpha}(\mathbb{W}_{\tau, \eta}^Q)$ , and conclude that there must be some axiom  $a \in j_{0, \alpha}(\mathcal{A})$  such that  $\dot{x} \not\models a$ . This axiom must have the form  $a_{\lambda, \rho, E, \vec{\phi}}^{Q_\alpha}$  for some  $\lambda, \rho, E, \vec{\phi} \in Q_\alpha$ . Let us pick a condition  $q_\alpha \in \mathbb{A}$  forcing this, and forcing value for the unsatisfied axiom  $a$ , say  $q$  forces  $\dot{x} \not\models a_{\lambda_\alpha, \rho_\alpha, E_\alpha, \vec{\phi}^\alpha}$ . Pick  $q_\alpha$  so that  $\rho_\alpha$  is minimal. We extend the tree by setting  $Q_{\alpha+1} = \text{Ult}(Q_\alpha, E_\alpha)$  for  $\alpha'$  least so that  $\text{crit } E_\alpha = \lambda_\alpha < \rho_{\alpha'}$ .

The genericity iteration Theorem will be proved by showing that the second case in the construction cannot hold for all  $\alpha < (\kappa^+)^V$ . This is very similar to the usual proof that comparisons of mice of size  $\kappa$  must terminate before reaching  $\kappa^+$ . Assume for contradiction that the construction continues to  $(\kappa^+)^V$ , and let  $\mathcal{T}$  be the tree of length  $(\kappa^+)^V$  constructed. Since  $Q$  is assumed to be  $(\kappa^+)^V + 1$ -iterable there exists a cofinal branch through the tree. Let  $b$  denote this branch. Note that  $b \subset (\kappa^+)^V$  is closed-unbounded.

For every  $\alpha \in b$  let  $\alpha_b^+$  be the least ordinal such that  $\alpha T \alpha_b^+ + 1$ . Then  $E_{\alpha_b^+}$  has critical point  $(\lambda_{\alpha_b^+})$  below  $\rho_\alpha$ , and is applied to  $Q_\alpha$  in the tree  $\mathcal{T}$  to form the ultrapower  $Q_{\alpha_b^+ + 1}$ . Note that  $\vec{\phi}^{\alpha_b^+}$  is in  $V_{\lambda_{\alpha_b^+ + 1}}^{Q_{\alpha_b^+}}$ , and since  $Q_\alpha$  and  $Q_{\alpha_b^+}$  agree on subsets of  $\lambda_{\alpha_b^+}$  it follows that  $\vec{\phi}^{\alpha_b^+} \in Q_\alpha$ . Let us denote  $\vec{\phi}^{\alpha_b^+}$  by  $\vec{\psi}^\alpha$ .

Let  $S_1 \subset b$  be the set of limit points of  $b$ . For  $\alpha \in S_1$  the model  $Q_\alpha$  is a direct limit and so  $Q_\alpha = \bigcup_{\beta < \alpha, \beta \in b} j''_{\beta, \alpha} Q_\beta$ . As  $\vec{\psi}^\alpha \in Q_\alpha$  there must exist some  $h(\alpha) < \alpha$  such that  $\vec{\psi}^\alpha \in j''_{h(\alpha), \alpha} Q_{h(\alpha)}$ . A standard pressing down argument now produces  $\beta < \kappa^+$  and stationary

$S_2 \subset S_1$  so that  $h(\alpha) = \beta$  for all  $\alpha \in S_2$ . Since  $Q_\beta$  has cardinality  $\kappa$ , further thinning of  $S_2$  produces stationary  $S_3 \subset S_2$  and a fixed  $\varphi \in Q_\beta$  such that  $\vec{\psi}^\alpha = j_{\beta,\alpha}(\vec{\varphi})$  for all  $\alpha \in S_3$ . Since  $\mathbb{A}$  too has cardinality  $\kappa$  we may assume further that for some fixed  $q \in \mathbb{A}$  we have  $q_{\alpha_b^+} = q$  for all  $\alpha \in S_3$ .

Let  $\alpha$  be any element of  $S_3$ , and let  $\gamma$  be  $\alpha_b^+$  (so  $\gamma+1 \in b$ ). Now  $q$  forces the real  $\dot{x}$  to contradict the axiom  $a_{\lambda_\gamma, \rho_\gamma, E_\gamma, j_{\beta,\alpha}(\vec{\varphi})}^{Q_\gamma}$ . This means that necessarily ( $q$  forces)  $\dot{x} \not\models \bigvee_{\xi < \lambda_\gamma} j_{\beta,\alpha}(\vec{\varphi})_\xi$  (\*), and  $\dot{x} \models \bigvee_{\xi < \nu_\gamma} i_{E_\gamma}^{Q_\gamma}(j_{\beta,\alpha}(\vec{\varphi}))_\xi$ . But  $i_{E_\gamma}^{Q_\gamma}(j_{\beta,\alpha}(\vec{\varphi})) = i_{E_\gamma}^{Q_\alpha}(j_{\beta,\alpha}(\vec{\varphi}))$ <sup>7</sup> since  $j_{\beta,\alpha}(\vec{\varphi})$  is an element of  $V_{\lambda_{\gamma+1}}^{Q_\alpha}$ . Thus  $\dot{x} \models \bigvee_{\xi < \nu_\gamma} i_{E_\gamma}^{Q_\alpha}(j_{\beta,\alpha}(\vec{\varphi}))_\xi$ .  $i_{E_\gamma}^{Q_\alpha}$  is simply  $j_{\alpha,\gamma+1}$ , so we can rewrite the above as ( $q$  forces)  $\dot{x} \models \bigvee_{\xi < \nu_\gamma} j_{\beta,\gamma+1}(\vec{\varphi})_\xi$ .

Consider now any  $\alpha' \in b$  such that  $\alpha' > \gamma+1$ . Then  $\text{crit}(j_{\gamma+1,\alpha'}) \geq \rho_\gamma$  (it is to secure this fact that we are forced to use iteration trees, and cannot manage with the simpler linear iterations), and so for  $\xi < \nu_\gamma$ ,  $j_{\beta,\gamma+1}(\vec{\varphi})_\xi$  is not moved by  $j_{\gamma+1,\alpha'}$ . Thus ( $q$  forces)  $\dot{x} \models \bigvee_{\xi < \nu_\gamma} j_{\beta,\alpha'}(\vec{\varphi})_\xi$ . But then clearly  $\dot{x} \models \bigvee_{\xi < \lambda_{\alpha_b^+}} j_{\beta,\alpha'}(\vec{\varphi})_\xi$ , and we now obtain a contradiction to (\*) by taking  $\alpha' \in S_3$ .

This concludes the proof of the first part of the second genericity Theorem. We leave the second half to the reader, and indicate here only how to define the forcing  $\dot{\mathbb{W}}_{\tau,\eta}^{Q,\mathbb{O}}$  when  $\mathbb{O}$  is a forcing notion in  $V_\tau^Q$ . Working in  $Q^\mathbb{O}$ , again consider the algebra of all formulae obtained from “ $n \in \tilde{x}$ ” closing under negations and wellordered disjunctions (in  $Q^\mathbb{O}$ ) of length  $< \eta$ . Let  $\dot{\mathcal{B}}$  be the set of axioms (computed in  $Q^\mathbb{O}$ )  $a_{\tilde{\lambda}, \tilde{\rho}, \dot{E}, \dot{\phi}}$  as before, with the restriction that  $\dot{E}$  must be an extender (of  $Q^\mathbb{O}$ ) induced by an extender of  $Q$ . I.e., there must exist an extender  $F \in Q$  such that the embedding  $i_E^{Q^\mathbb{O}}: Q^\mathbb{O} \rightarrow \text{Ult}(Q^\mathbb{O}, \dot{E})$  extends the embedding  $i_F^Q: Q \rightarrow \text{Ult}(Q, F)$ . Set then

$$\dot{\phi} \approx \dot{\psi} \text{ iff } \dot{\mathcal{B}} \vdash \dot{\phi} \leftrightarrow \dot{\psi}; \text{ and } \dot{\phi} \leq \dot{\psi} \text{ iff } \dot{\mathcal{B}} \vdash \dot{\phi} \rightarrow \dot{\psi}.$$

$\dot{\mathbb{W}}_{\tau,\eta}^{Q,\mathbb{O}}$  is then defined to be the set of equivalence classes of  $\approx$ , ordered by  $\leq$ . The proof of the genericity Theorem proceeds as before. The reader can verify this, noting that there are many extenders  $\dot{E}$  in  $Q^\mathbb{O}$  which are induced by extenders in  $Q$  — in fact there are enough such extenders to witness that  $\eta$  is a Woodin cardinal (because  $\mathbb{O}$  is a “small” forcing). This allows carrying out the argument of the Claim above, and subsequently the rest of the proof.

## References

- [FM95] M. Foreman and M. Magidor. Large cardinals and definable counter examples to the continuum hypothesis. *Ann. of Pure and Appl. Logic*, 76:47–97, 1995.
- [HMW] Kai Hauser, Adrian D.R. Mathias, and W. Hugh Woodin. *The axiom of determinacy*. Forthcoming.
- [Jec78] Thomas Jech. *Set Theory*. Academic Press, 1978.
- [Jec87] Thomas Jech. *Multiple forcing*. Cambridge University Press, 1987.

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<sup>7</sup>We replaced  $Q_\gamma$  with  $Q_\alpha$ .

- [MS94] D.A. Martin and John Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.
- [NZ98] I. Neeman and J. Zapletal. Proper forcing and absoluteness in  $L(\mathbb{R})$ . *Comment. Math. Univ. Carolinae*, 39(2):281–301, 1998.
- [Ste93] John Steel. Inner models with many Woodin cardinals. *Annals of pure and applied logic*, 65(2):185–209, 1993.
- [Woo99] W.H. Woodin. *The axiom of determinacy, forcing axioms, and the non-stationary ideal*. de Gruyter, 1999.